

Gravity effects of the quantum vacuum. Dark energy and dark matter

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May, 11, 2015

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Abstract

The stress-energy tensor of the quantum vacuum is studied for the particular case of quantum electrodynamics (QED), that is a fictitious universe where only the electromagnetic and the electron-positron fields exist. The integrals involved are ultraviolet divergent but it is suggested that a natural cut-off may exist. It is shown that, in spite of the fact that the stress-energy tensor of the electromagnetic field alone is traceless (i.e the pressure P equals $1/3$ the energy density u), the total QED tensor is proportional to the metric tensor to a good approximation (i. e. $P = -u$). It is proposed that there is a cosmological constant in Einstein equation that exactly balances the stress-energy of the vacuum. It is shown that vacuum fluctuations give rise to a modified spacetime metric able to explain dark energy. Particular excitations of the vacuum are studied that might explain dark matter.

1 Introduction

1.1 Quantum vacuum, cosmological constant and dark energy

Quantum field theory predicts that the vacuum is not empty, but filled with interacting quantum fields. A calculation of the energy of the fields gives divergent quantities, positive for free Bose fields and negative for free Fermi fields, as will be illustrated in section 2 below for the particular case of quantum electrodynamics (QED). Nevertheless the energy of the vacuum is irrelevant in most quantum calculations not including gravity. In fact it may be ignored defining the zero of energy at the level of the vacuum. In practice this is achieved using the “normal ordering rule” for the creation and annihilation field operators.

However a big difficulty appears if we want to take the vacuum gravity into account, because in this case the zero of energy cannot be fixed at will. Two alternative solutions have been proposed, none of them completely satisfactory: 1) Assuming that the vacuum fields represent a mathematical artefact of the quantum formalism and only the excitations of the fields above the vacuum contribute actual energy, 2) The vacuum fields have energy,

but positive and negative contributions may cancel to each other, at least approximately. In this paper the first possibility is rejected and the second one is studied searching for clues for the solution of a number of well known difficulties.

If there is an actual energy density of the vacuum there should be also pressure terms, that is a full stress-energy tensor. Indeed if the spacetime background is Minkowski, at least approximately, then it seems that the vacuum properties should be Lorentz invariant whence the vacuum stress-energy tensor would be proportional to the metric tensor. As a consequence the effect of that tensor would be equivalent to a cosmological term in the Einstein equation of general relativity. The existence of such a term might explain the observations in cosmology, which have shown that the universe is in accelerated expansion[1][2]. Actually the cause of that expansion is unknown and it is usually named “dark energy”, but it is common wisdom to interpret dark energy as an effect of the quantum vacuum. Other alternatives have been proposed that will not be commented here.

The hypothesis that dark energy is an effect of the quantum vacuum poses a well known problem[4]. In fact it is plausible that the vacuum energy density, ρ_{DE} , should correspond to a combination of the universal constants c, \hbar, G , that is

$$\rho_{DE} \sim \frac{c^5}{G^2 \hbar} \simeq 10^{97} kg/m^3. \quad (1)$$

This is Planck’s density whose value is about 123 orders greater than the required dark energy density, with current empirical value[3]

$$\rho_{DE} = -P_{DE} \simeq (6.0 \pm 0.2) \times 10^{-27} kg/m^3, \quad (2)$$

$P_{DE} < 0$ being the pressure. The first eq.(2) is consistent with the vacuum stress-energy tensor being Lorentz invariant.

1.2 The relevance of the vacuum fluctuations

In this paper a solution is proposed to the problem of the big disagreement between the “theoretical” value eq.(1), and the empirical result eq.(2). In order to understand the logic of the proposal I start pointing out the relevance of the fluctuations of the vacuum fields, that I believe has not been fully appreciated in the analysis of the gravitational effects of the vacuum. For

the sake of clarity I will begin with arguments involving classical (rather than quantum) fields and Newtonian gravity (rather than general relativity).

Let us assume that the vacuum consists of a stationary, homogeneous and isotropic set of interacting fields, some of the fields contributing positive energy and other fields negative energy and similarly for the interactions. If we want to escape from the absurd assumption that the field energy density is at the Planck scale (i. e. eq.(1)) a plausible hypothesis is that positive and negative contributions cancel to each other. A partial cancelation giving the result eq.(2), that is more than hundred orders smaller than Planck's density, looks conspiratory[4]. Therefore it is plausible that the cancelation is exact, which I will assume at this moment (this assumption will be later modified, taking quantum theory and general relativity into account, see section 4 below). If this is the case it seems that dark energy remains unexplained. However I will argue in the following that, even if the average energy density of the vacuum is zero, the fluctuations might explain dark energy.

In fact let us suppose that the vacuum fields fluctuate. That is, at a given time there are regions with energy density above the mean (that is positive fluctuations) and other regions with density below it (negative fluctuations). Thus, assuming that the mean vacuum energy is zero, the fluctuation of the density would be a function, $\rho(\mathbf{r},t)$, fulfilling the condition that its space average is zero, that is

$$\lim_{V \rightarrow \infty} \int_V \rho(\mathbf{r},t) d^3\mathbf{r} = 0. \quad (3)$$

We must assume, as in special relativity, that energy gravitates. Then we should accept that Newtonian gravity associates negative (positive) gravitational potential to positive (negative) mass-energy. (Of course Newtonian gravity is not consistent with special relativity, but for a simplified argument I may combine both). Thus, using Newtonian theory, the gravitational energy, E_G , of the vacuum fluctuations in the volume V would be

$$\begin{aligned} E_G &= \frac{1}{2} \int_V \rho(\mathbf{r}_1,t) \Phi(\mathbf{r}_1,t) d^3\mathbf{r}_1, \\ \Phi(\mathbf{r}_1,t) &\equiv -G \int_V \frac{\rho(\mathbf{r}_2,t)}{|\mathbf{r}_2 - \mathbf{r}_1|} d^3\mathbf{r}_2, \end{aligned} \quad (4)$$

where G is Newton's constant and V a volume large in comparison with the range of the fluctuations. As the fluctuating density ρ has zero mean,

the gravitational potential of the fluctuations Φ , which is linearly related to ρ , is also nil on the average. Thus the vacuum fluctuations would produce no average gravitational field at any spacetime point. This might suggest that vacuum fluctuations would not lead to spacetime curvature when we pass from Newton's to Einstein's gravity. If this is the case the density fluctuations could not explain dark energy. However the suggestion is flawed due to the fact that the equations of Einstein's theory are nonlinear, in contrast with Newton's, a point that will be explained in more detail in section 4 below.

There is another argument supporting that vacuum fluctuations, with zero mean, may produce a long range effect. In fact, the gravitational energy E_G due to the fluctuations, eq.(4), is not linear but quadratic in the fluctuating density. Therefore the mean gravitational energy density, $\bar{\rho}_G$, needs not be zero.

The quantity $\bar{\rho}_G$ is defined by

$$\begin{aligned}\bar{\rho}_G &= \lim_{V \rightarrow \infty} \frac{E_G}{V} \\ &= - \lim_{V \rightarrow \infty} \left[\frac{G}{2V} \int_V \rho(\mathbf{r}_1, t) d^3\mathbf{r}_1 \int_V |\mathbf{r}_2 - \mathbf{r}_1|^{-1} \rho(\mathbf{r}_2, t) d^3\mathbf{r}_2 \right].\end{aligned}$$

Changing from the variables $\{\mathbf{r}_1, \mathbf{r}_2\}$ to the new ones $\{\mathbf{r}_0, \mathbf{r}\}$ via

$$\mathbf{r}_0 = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1,$$

we get

$$\begin{aligned}\bar{\rho}_G &= -\frac{G}{2} \int \frac{1}{r} C(r) d^3\mathbf{r} = -2\pi G \int_0^\infty C(r) r dr, \\ C(r) &\equiv \lim_{V \rightarrow \infty} \left[\frac{1}{V} \int_V \rho(\mathbf{r}_0 + \mathbf{r}/2, t) \rho(\mathbf{r}_0 - \mathbf{r}/2, t) d^3\mathbf{r}_0 \right].\end{aligned}\tag{5}$$

We see that the mean gravitational energy of the fluctuations depends on the two-point energy density correlation, $C(r)$, and it may be different from zero. The two-point correlation at equal times should depend only on the distance $r = |\mathbf{r}_2 - \mathbf{r}_1|$, as is consistent with the assumed homogeneity and isotropy of the vacuum. Also it fulfils the condition

$$\begin{aligned}\int C(r) d^3\mathbf{r} &= \lim_{V \rightarrow \infty} \left[\frac{1}{V} \int_V d^3\mathbf{r} \int_V \rho(\mathbf{r}_0 + \mathbf{r}/2, t) \rho(\mathbf{r}_0 - \mathbf{r}/2, t) d^3\mathbf{r}_0 \right] = \\ &= \lim_{V \rightarrow \infty} \left[\frac{1}{V} \int_V \rho(\mathbf{r}_1, t) d^3\mathbf{r}_1 \times \int_V \rho(\mathbf{r}_2, t) d^3\mathbf{r}_2 \right] = 0,\end{aligned}$$

taking eq.(3) into account.

The conclusion from our classical analysis is that, in addition to the mean energy density of the vacuum, if any, there should be a gravitational energy density caused by the density fluctuations. In section 3 I will calculate the two-point correlation in quantum electrodynamics. In section 4 the relation of this correlation with the dark energy density ρ_{DE} , eq.(2), will be studied. Indeed we shall see that $\bar{\rho}_G$ may be taken as an estimate for ρ_{DE} .

The suggestion that quantum vacuum fluctuations may give rise to an effective cosmological constant was made by Zeldovich[5] in 1976. The possibility that these fluctuations are at the origin of the dark energy has been explored recently[6] using a simplified model involving general relativity.

2 Stress-energy tensor of the vacuum fields

2.1 Energy density and pressure. Mean and two-point correlations

Our aim is to study the gravitational effects of the vacuum. According to general relativity the quantity relevant for that purpose is the stress-energy tensor. If we assume that space-time is Minkowski (which is a good approximation for our present purposes, but see section 3) then stationarity, isotropy and homogeneity of the vacuum lead to a metric tensor defined by just two quantities, the energy density ρ and the pressure P . That is the metric tensor should be

$$T_1^1 = T_2^2 = T_3^3 = P, T_0^0 = -\rho, T_\mu^\nu = 0 \text{ if } \mu \neq \nu.$$

If in addition we assume Lorentz invariance, then we should have $P = -\rho$. In this case the stress-energy tensor is proportional to the metric tensor. Actually I shall not assume Lorentz invariance from the start, but should derive it as a result of the calculations. In order to obtain the gravitational contribution of the vacuum it is enough to get the mean energy density, the mean pressure and the relevant two-point correlation functions. The average energy and pressure will be calculated in section 2 and some of the correlations in section 3.

In the quantum context vacuum expectations of the appropriate operators should be substituted for averages. The expectation of the energy density is

$$\rho = \langle vac | \hat{\rho}(\mathbf{r}, t) | vac \rangle. \quad (6)$$

where the operator $\hat{\rho}(\mathbf{r}, t)$ is the quantized Hamiltonian density, which should include all quantum fields and their interactions, excluding the gravitational one. In this paper the gravitational field (that is the spacetime curvature) is considered as radically different from the remaining fields of nature, although it should be quantized in some form (see below, section 3.2). In the Heisenberg picture the Hamiltonian (or energy) density operator may depend on position and time, but the state should not. The expectation ρ , eq.(6), cannot depend on (\mathbf{r}, t) due to the translational and rotational invariance of the vacuum.

The two-point density correlation is the most relevant quantity for the study of the quantum fluctuations. It is defined in quantum field theory by

$$C(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle vac | \hat{\rho}(\mathbf{r}_1, t_1) \hat{\rho}(\mathbf{r}_2, t_2) | vac \rangle, \quad (7)$$

provided that $\hat{\rho}(\mathbf{r}_1, t_1)$ commutes with $\hat{\rho}(\mathbf{r}_2, t_2)$. If they do not commute the correlation so defined might not be real, having an imaginary part. If this is the case we will define the correlation in the form

$$C = \frac{1}{2} \langle vac | \hat{\rho}(\mathbf{r}_1, t_1) \hat{\rho}(\mathbf{r}_2, t_2) + \hat{\rho}(\mathbf{r}_2, t_2) \hat{\rho}(\mathbf{r}_1, t_1) | vac \rangle. \quad (8)$$

If we integrate the Hamiltonian density operator, $\hat{\rho}(\mathbf{r}, t)$, with respect to \mathbf{r} we get the Hamiltonian operator

$$\hat{H} = \int_V \hat{\rho}(\mathbf{r}, t) d^3r. \quad (9)$$

I will consider the volume V as finite (e. g. a cube of side $V^{1/3}$), but take the limit V becoming the whole space at the end of the calculations. I shall assume that the vacuum state is the eigenstate of \hat{H} with the minimal eigenvalue, which I will label E_{vac} . That is

$$\hat{H} | vac \rangle = E_{vac} | vac \rangle, E_{vac} = V \rho_{vac},$$

with no other eigenvalue of \hat{H} smaller than E_{vac} . Hence the \mathbf{r}_2 integral of the two-point correlation gives

$$\begin{aligned} \int_V C(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) d^3r_2 &= \left\langle vac \left| \hat{\rho}(\mathbf{r}_1, t_1) \int_V \hat{\rho}(\mathbf{r}_2, t_2) d^3r_2 \right| vac \right\rangle \\ &= \left\langle vac \left| \hat{\rho}(\mathbf{r}_1, t_1) \hat{H} \right| vac \right\rangle \\ &= \langle vac | \hat{\rho}(\mathbf{r}_1, t_1) E_{vac} | vac \rangle = V \rho_{vac}^2, \end{aligned} \quad (10)$$

where eqs.(7) and (9) have been taken into account. This equation contradicts eq.(??), the reason being that here we have used a different definition, not making the assumption $\rho_{vac} = 0$. Actually the relevant quantity is the two-point *density correlation relative to the mean*, therefore I will redefine the correlation after subtracting the square of the mean density, that is

$$C(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = C_{old}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) - \rho_{vac}^2. \quad (11)$$

where C_{old} is given by eq.(8). This correlation does fulfil eq.(??).

A calculation of the energy density and the two-point correlation of all quantum fields existing in nature (possibly those of the standard model of high energy physics) is a formidable task, out of the scope of this paper. Here the study will be restricted to quantum electrodynamics (QED) in order to get clues for the solution of the problems associated with the gravity of the vacuum fields. That is I will consider a fictitious world where the existing quantum fields are the electromagnetic one and the Dirac field of electrons and positrons but no other fields.

2.2 The free electromagnetic field. A paradox

We will work in Minkowski space, a good enough approximation for the calculation of the mean vacuum energy density. (But the approximation is not valid when we take the two-point correlation into account, as will be discussed in section 3.2). In the present subsection I will calculate the stress-energy tensor of the free electromagnetic field of the vacuum, ignoring all other vacuum fields. I shall assume homogeneity and isotropy, so that the stress-energy tensor is defined by just two parameters, density ρ and pressure P . However, as we shall see, it is not possible to assume Lorentz invariance, that is $P = -\rho$.

The quantization of a classical field involves promoting the amplitudes of the plane waves expansion to become (creation or annihilation) operators. For instance the classical free electromagnetic field, in the Coulomb gauge, may be represented by an expansion in plane waves of the vector potential, $\mathbf{A}(\mathbf{r}, t)$, that is

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) = & \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, \epsilon} \left(\frac{\hbar}{2k} \right)^{1/2} \times \\ & \times [\alpha_{\mathbf{k}, \epsilon} \boldsymbol{\epsilon} \exp(i\mathbf{k} \cdot \mathbf{r} - ikt) + \alpha_{\mathbf{k}, \epsilon}^* \boldsymbol{\epsilon} \exp(-i\mathbf{k} \cdot \mathbf{r} + ikt)], \end{aligned} \quad (12)$$

where $k = |\mathbf{k}|$. From now on I will use the notation of the book of Sakurai[7] (except otherwise stated) and natural units $\hbar = c = 1$. However sometimes I will write explicitly Planck's constant \hbar for clarity. From eq.(12) it is easy to get the electric field, $\mathbf{E} = -\partial\mathbf{A}/\partial t$, and the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$. The polarization vector $\boldsymbol{\varepsilon}$ depends on \mathbf{k} (in fact $\mathbf{k} \cdot \boldsymbol{\varepsilon} = 0$) and it may have two possible values so that we should write $\boldsymbol{\varepsilon}_j(\mathbf{k})$, $j = 1, 2$, but I will use a simplified notation except when some confusion might arise.

In the quantized field an annihilation operator $\hat{\alpha}_{\mathbf{k},\boldsymbol{\varepsilon}}$ is substituted for the amplitude $\alpha_{\mathbf{k},\boldsymbol{\varepsilon}}$ and a creation operator $\hat{\alpha}_{\mathbf{k},\boldsymbol{\varepsilon}}^\dagger$, for the amplitude $\alpha_{\mathbf{k},\boldsymbol{\varepsilon}}^*$, whence the electric and magnetic fields, $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$, become vector operators. They may be written as an expansion in plane waves taking a quantized counterpart of eq.(12) into account. From these expansions, that I do not write explicitly, it is trivial to obtain the energy density operator. We get

$$\begin{aligned} \hat{\rho}_{EM} &\equiv \frac{1}{2} (\hat{\mathbf{E}}^2 + \hat{\mathbf{B}}^2) = \frac{1}{4V} \sum_{\mathbf{k},\boldsymbol{\varepsilon}} \sum_{\mathbf{k}',\boldsymbol{\varepsilon}'} K_1 (\hat{\alpha}_{\mathbf{k}',\boldsymbol{\varepsilon}'}^\dagger \hat{\alpha}_{\mathbf{k},\boldsymbol{\varepsilon}} + \hat{\alpha}_{\mathbf{k},\boldsymbol{\varepsilon}} \hat{\alpha}_{\mathbf{k}',\boldsymbol{\varepsilon}'}^\dagger) \\ &\quad + \frac{1}{4V} \sum_{\mathbf{k},\boldsymbol{\varepsilon}} \sum_{\mathbf{k}',\boldsymbol{\varepsilon}'} \left(K_2 \hat{\alpha}_{\mathbf{k},\boldsymbol{\varepsilon}} \hat{\alpha}_{\mathbf{k}',\boldsymbol{\varepsilon}'} + K_2^* \hat{\alpha}_{\mathbf{k},\boldsymbol{\varepsilon}}^\dagger \hat{\alpha}_{\mathbf{k}',\boldsymbol{\varepsilon}'}^\dagger \right), \end{aligned}$$

where the functions K_1 and K_2 are c-numbers (not operators) given by

$$K_1 = \sqrt{k k'} \boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}' \exp [i (\mathbf{k} - \mathbf{k}') \cdot \mathbf{r} - i (k - k') t] \quad (13)$$

$$K_2 = \sqrt{k k'} \boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}' \exp [i (\mathbf{k} + \mathbf{k}') \cdot \mathbf{r} - i (k + k') t], \quad (14)$$

For notational simplicity I have introduced the following “star product”

$$\boldsymbol{\varepsilon} * \boldsymbol{\varepsilon}' \equiv \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}' + \frac{1}{k k'} [(\mathbf{k} \times \boldsymbol{\varepsilon}) \cdot (\mathbf{k}' \times \boldsymbol{\varepsilon}')],$$

with the properties

$$\begin{aligned} \sum_j (\boldsymbol{\varepsilon}_i * \boldsymbol{\varepsilon}'_j) (\boldsymbol{\varepsilon}'_j * \boldsymbol{\varepsilon}''_l) &= \boldsymbol{\varepsilon}_i * \boldsymbol{\varepsilon}''_l, \quad \sum_{ij} \boldsymbol{\varepsilon}_i * \boldsymbol{\varepsilon}_j = 4 \\ \sum_{ij} (\boldsymbol{\varepsilon}_i * \boldsymbol{\varepsilon}'_j)^2 &= 2 \left[1 + \frac{\mathbf{k} \cdot \mathbf{k}'}{k k'} \right]^2. \end{aligned} \quad (15)$$

It is convenient to write all terms of $\hat{\rho}_{EM}$ in normal order, that is the annihilation (creation) operators to the right (left). Taking the commutation

rules of the field operators into account we obtain

$$\begin{aligned}
\hat{\rho}_{EM} &= \rho_{EM0} + \hat{\rho}_{EM1} + \hat{\rho}_{EM2}, \\
\hat{\rho}_{EM1} &= \frac{1}{2V} \sum_{\mathbf{k}, \epsilon} \sum_{\mathbf{k}', \epsilon'} K_1 \hat{\alpha}_{\mathbf{k}', \epsilon'}^\dagger \hat{\alpha}_{\mathbf{k}, \epsilon}, \\
\hat{\rho}_{EM2} &= \frac{1}{4V} \sum_{\mathbf{k}, \epsilon} \sum_{\mathbf{k}', \epsilon'} \left(K_2 \hat{\alpha}_{\mathbf{k}, \epsilon} \hat{\alpha}_{\mathbf{k}', \epsilon'} + K_2^* \hat{\alpha}_{\mathbf{k}, \epsilon}^\dagger \hat{\alpha}_{\mathbf{k}', \epsilon'}^\dagger \right),
\end{aligned} \tag{16}$$

where ρ_{EM0} is a numerical constant (times de unit operator) given by

$$\rho_{EM0} = \frac{1}{2V} \sum_{\mathbf{k}, \epsilon} k.$$

The Hamiltonian is obtained by performing a space integral of the energy density eq.(16), that is

$$\begin{aligned}
\hat{H}_{EM} &= \lim_{V \rightarrow \infty} \int_V \hat{\rho}_{EM}(\mathbf{r}) d^3\mathbf{r} \\
&= \sum_{\mathbf{k}, \epsilon} k (\hat{\alpha}_{\mathbf{k}, \epsilon}^\dagger \hat{\alpha}_{\mathbf{k}, \epsilon} + \frac{1}{2}).
\end{aligned} \tag{17}$$

We see that the integral cancels de V denominator and removes the spacetime dependence.

For the free electromagnetic field the vacuum state, $|0\rangle$, may be defined as the state with the minimal energy, that is the smallest eigenvector of the operator eq.(17). It is a state with zero photons and it has the properties

$$\alpha_{\mathbf{k}, \epsilon} |0\rangle = 0, \langle 0 | \alpha_{\mathbf{k}, \epsilon}^\dagger = 0.$$

The state $|0\rangle$ is different from the actual QED vacuum, $|vac\rangle$, that takes the interaction with the electron-positron field into account. It will be studied below. For the purely electromagnetic vacuum state $|0\rangle$ the expectation of the energy density is

$$\langle 0 | \hat{\rho}_{EM} | 0 \rangle = \rho_{EM0} = \frac{1}{V} \sum_{\mathbf{k}, \epsilon} \frac{1}{2} \hbar k = \frac{1}{V} \sum_{\mathbf{k}} \hbar k, \tag{18}$$

where the latter equality derives from the two possible polarizations. We have taken into account that the two latter terms of eq.(16) do not contribute

to the expectation value because either the annihilation operator placed on the right or the creation operator on the left gives zero when acting on the vacuum state. In the limit $V \rightarrow \infty$ we obtain

$$\begin{aligned}\rho_{EM0} &= \frac{1}{V} \sum_{\mathbf{k}} k \rightarrow \int k (2\pi)^{-3} d^3k, \\ &= \frac{1}{2\pi^2} \int_0^{k_{\max}} k^3 dk = \frac{k_{\max}^4}{8\pi^2},\end{aligned}\tag{19}$$

where we have introduced a cut-off, k_{\max}^4 , in the wavevectors (or the momenta, what is equivalent in our units). We see that the integral is strongly divergent in the limit $k_{\max} \rightarrow \infty$, but it is plausible that a cut-off could be originated by fluctuations of the spacetime metric, as will be discussed in section 3.2.

In order to define the stress-energy tensor we need the pressure. As is well known the pressure is 1/3 times the energy density for any isotropic electromagnetic radiation. Therefore the pressure of the vacuum electromagnetic field would be one third the quantity eq.(19). Hence the stress-energy tensor of the free electromagnetic field of the vacuum is diagonal, and its nonzero components are

$$\rho_{EM0} = \frac{k_{\max}^4}{8\pi^2}, P_{EM0} = \frac{k_{\max}^4}{24\pi^2},\tag{20}$$

That is the tensor is traceless, fulfilling

$$T^\mu_\mu = 3P_{EM0} - \rho_{EM0} = 0,$$

for any choice of the cut-off wavevector k_{\max} .

This gives rise to a paradox because it is plausible that the stress-energy tensor of the vacuum is proportional to the metric tensor, as a consequence of the Lorentz invariance that we should assume for the vacuum (in Minkowski space). Equivalently the tensor in mixed coordinates should be diagonal with a pressure equal to *minus* the energy density. Therefore if the vacuum fields are real, which is the fundamental hypothesis of this paper, there is a contradiction between the stress-energy tensor predicted by the Maxwell equations (for any free radiation field) and the Lorentz invariance of the vacuum fields. Actually it may be argued that neither the density nor the pressure are well defined, eqs.(20) being divergent (or cut-off dependent), but there is a problem in any case. I think that the plausible solution is that, as

the vacuum contains many fields, *the stress-energy tensor of all interacting fields together is well defined and Lorentz invariant*, but the tensor of every free field alone is not. (The divergences might be removed by an effective cut-off, see section 3.2). Studying the stress-energy tensor of all fields is beyond the scope of this paper, but a support to our hypothesis results from the study of the electron-positron field to be made in the next subsection.

2.3 The electron-positron vacuum field

We need the stress-energy tensor of the Dirac field, that is the energy density, ρ , and the pressure, P . In order to avoid any confusion caused by the different notations used in the literature, I will write ρ and P in terms of the original Dirac matrices α_k and β (rather than the gamma matrices that have been defined in several different forms, compare e. g. [7] and [8]). Thus the corresponding operators may be written

$$\hat{\rho}_D = \frac{i}{2} \left(\hat{\psi}^\dagger \frac{d\hat{\psi}}{dt} - \frac{d\hat{\psi}^\dagger}{dt} \hat{\psi} \right), \quad (21)$$

$$\hat{P}_D^{(k)} = \frac{i}{2} \left(\hat{\psi}^\dagger \alpha_j \frac{\partial \hat{\psi}}{\partial x^j} - \frac{\partial \hat{\psi}^\dagger}{\partial x^j} \alpha_j \hat{\psi} \right), \quad (22)$$

where $\hat{\psi}$ and $\hat{\psi}^\dagger$ are quantized fields and j may be either 1, 2 or 3, the resulting pressure being the same in the three cases due to the assumed isotropy of the vacuum. Expanding $\hat{\psi}$ and $\hat{\psi}^\dagger$ in plane waves we get

$$\begin{aligned} \hat{\psi}(\mathbf{r}, t) &= \sqrt{\frac{1}{V}} \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E}} \hat{b}_{\mathbf{p}s} u_{\mathbf{p}s} \exp(i\mathbf{p} \cdot \mathbf{r} - iEt) \\ &\quad + \sqrt{\frac{1}{V}} \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E}} \hat{d}_{\mathbf{p}s}^\dagger v_{\mathbf{p}s} \exp(-i\mathbf{p} \cdot \mathbf{r} + iEt), \\ \hat{\psi}^\dagger(\mathbf{r}, t) &= \sqrt{\frac{1}{V}} \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E}} \hat{b}_{\mathbf{p}s}^\dagger u_{\mathbf{p}s}^\dagger \exp(-i\mathbf{p} \cdot \mathbf{r} + iEt) \\ &\quad + \sqrt{\frac{1}{V}} \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E}} \hat{d}_{\mathbf{p}s} v_{\mathbf{p}s}^\dagger \exp(i\mathbf{p} \cdot \mathbf{r} - iEt), \end{aligned}$$

where $\hat{b}_{\mathbf{p},s}$ ($\hat{d}_{\mathbf{p},s}$) is the annihilation operator of an electron (positron) with momentum \mathbf{p} and spin $s(= 1, 2)$, and $\hat{b}_{\mathbf{p},s}^\dagger$ ($\hat{d}_{\mathbf{p},s}^\dagger$) the corresponding creation operator and $u, u^\dagger, v, v^\dagger$ are appropriate spinors. (Our notation follows the book of Sakurai[7].) Inserting these expressions in eq.(21) we get $\hat{\rho}_D$ as a sum of 4 terms with a product of 2 operators each. We want all the terms with the creation and annihilation operators in normal order so that, taking the anticommutation rules into account, we should make a replacement as follows

$$\begin{aligned} & \hat{d}_{\mathbf{p}s} \exp(i\mathbf{p} \cdot \mathbf{r} - iEt) \hat{d}_{\mathbf{p}'s'}^\dagger \exp(-i\mathbf{p}' \cdot \mathbf{r} + iE't) \\ &= \delta_{\mathbf{p}\mathbf{p}'} \delta_{ss'} - \hat{d}_{\mathbf{p}'s'}^\dagger \exp(-i\mathbf{p}' \cdot \mathbf{r} + iE't) \hat{d}_{\mathbf{p}s} \exp(i\mathbf{p} \cdot \mathbf{r} - iEt), \end{aligned}$$

where $\delta_{\mathbf{p}\mathbf{p}'}$ and $\delta_{ss'}$ are Kronecker's deltas. Thus we obtain 5 terms, all in normal order, that is

$$\hat{\rho}_D(\mathbf{r}, t) = \rho_{D0} + \hat{\rho}_b(\mathbf{r}, t) + \hat{\rho}_d(\mathbf{r}, t) + \hat{\rho}_{bd}(\mathbf{r}, t) + \hat{\rho}_{bd}^\dagger(\mathbf{r}, t). \quad (23)$$

The latter 4 terms of eq.(23) are as follows

$$\begin{aligned} \hat{\rho}_b(\mathbf{r}, t) &= \frac{1}{V} \sum_{\mathbf{p}\mathbf{p}'ss'} B(\mathbf{p}, s, \mathbf{p}', s') \hat{b}_{\mathbf{p}'s'}^\dagger b_{\mathbf{p}s} \exp[i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{r} - i(E - E')t], \\ \hat{\rho}_d(\mathbf{r}, t) &= \frac{1}{V} \sum_{\mathbf{p}\mathbf{p}'ss'} D(\mathbf{p}, s, \mathbf{p}', s') d_{\mathbf{p}'s'}^\dagger d_{\mathbf{p}s} \exp[i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{r} - i(E - E')t] \\ \hat{\rho}_{bd}(\mathbf{r}, t) &= \frac{1}{V} \sum_{\mathbf{p}\mathbf{p}'ss'} F(\mathbf{p}, s, \mathbf{p}', s') d_{\mathbf{p}s} b_{\mathbf{p}'s'} \exp[i(\mathbf{p} + \mathbf{p}') \cdot \mathbf{r} - i(E + E')t], \\ \hat{\rho}_{bd}^\dagger(\mathbf{r}, t) &= \frac{1}{V} \sum_{\mathbf{p}\mathbf{p}'ss'} F^*(\mathbf{p}, s, \mathbf{p}', s') b_{\mathbf{p}'s'}^\dagger d_{\mathbf{p}s}^\dagger \exp[-i(\mathbf{p} + \mathbf{p}') \cdot \mathbf{r} + i(E + E')t]. \end{aligned}$$

Getting the functions B, D, F and F^* is straightforward but we report only the expression of $|F|^2$ after the spin sum, which will be used in section 3. We get

$$\begin{aligned} \sum_{ss'} |F|^2 &= \frac{m^2}{4EE'} (E - E')^2 \sum_{ss'} |v_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}')|^2 \\ &= \frac{m^2}{4} (E - E')^2 \left[1 + \frac{\mathbf{p} \cdot \mathbf{p}' - m^2}{EE'} \right], \end{aligned} \quad (24)$$

where we have taken into account the expressions of the spinors u and v [7]. This leads to

$$\sum_{ss'} |v_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{q})|^2 = \frac{(E+m)(E'+m)}{2m^2} \left| \frac{\mathbf{p}}{E+m} + \frac{\mathbf{q}}{E'+m} \right|^2, \quad (25)$$

whence putting

$$\begin{aligned} \mathbf{p}^2 &= E^2 - m^2 = (E+m)(E-m), \\ \mathbf{q}^2 &= E'^2 - m^2 = (E'+m)(E'-m), \end{aligned}$$

it is easy to get eq.(24).

The first term in eq.(23), ρ_{D0} , is a c-number, not an operator (more properly it is proportional to the unit operator), that is

$$\rho_{D0} = -\frac{1}{V} \sum_{p,s} \sqrt{m^2 + p^2}. \quad (26)$$

Integration of eq.(23) with respect to \mathbf{r} gives the Hamiltonian of the free electron-positron field, that is

$$H_D = \sum_{p,s} \sqrt{m^2 + p^2} (b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s} + d_{\mathbf{p},s}^\dagger d_{\mathbf{p},s} - 1). \quad (27)$$

The vacuum state, $|0\rangle$, of the free field corresponds to the eigenvector of the Hamiltonian eq.(27) with the smallest eigenvalue. It consists of zero electrons and zero positrons. Therefore from now on we will define $|0\rangle$ to be the QED unperturbed vacuum state, which is a simultaneous eigenvector of both Hamiltonians eqs.(17) and (27). Thus the state $|0\rangle$ is defined, from now on, as having zero photons, electrons and positrons. It should be distinguished from the physical vacuum state, $|vac\rangle$, which is an eigenvalue of the total Hamiltonian, including the interactions.

Our aim is to obtain the energy density, that is the expectation value in the vacuum of the density operator eq.(23). Neglecting the field interactions (to be studied in the next subsection), we get the energy density of the unperturbed electron-positron field as the vacuum expectation of the operator, that is

$$\langle 0 | \hat{\rho}_D(\mathbf{r}, t) | 0 \rangle = \langle 0 | \hat{\rho}_{D0}(\mathbf{r}, t) | 0 \rangle = \rho_{D0}$$

The former equality follows from the fact that the four latter terms of eq.(23) do not contribute to the expectation because either the annihilation operator placed on the right or the creation operator on the left gives zero when acting on the vacuum state.

The quantity ρ_{D0} , eq.(26), leads to the following *negative* divergent energy density in the limit $V \rightarrow \infty$

$$\begin{aligned}\rho_{D0} &= -(2\pi)^{-3} \sum_s \int_0^{p_{\max}} E d^3p = -\pi^{-2} \int_0^{p_{\max}} \sqrt{m^2 + p^2} p^2 dp \\ &= -\frac{1}{8\pi^2} \left[p_{\max} \sqrt{m^2 + p_{\max}^2} (m^2 + 2p_{\max}^2) - m^4 \arg \sinh \left(\frac{p_{\max}}{m} \right) \right] \\ &= -\frac{1}{4\pi^2} \left[p_{\max}^4 + p_{\max}^2 m^2 + \frac{1}{8} m^4 - \frac{1}{2} m^4 \ln \left(\frac{2p_{\max}}{m} \right) \right] + O(p_{\max}^{-2})\end{aligned}\quad (28)$$

where we have introduced an ultraviolet cut-off of the momenta, p_{\max} . I point out that the negative value might be anticipated by inspection of the Hamiltonian eq.(27). In fact the density ρ_{D0} is just the (unperturbed) vacuum expectation of the Hamiltonian divided by the volume V .

The stresses may be calculated as vacuum expectations of the operators $\hat{P}_D^{(k)}$, eq.(22). As in the case of the energy density, eq.(23), we may write the operator $\hat{P}_D^{(k)}$ as a sum of terms all of them with the creation and annihilation operators in normal order. We get

$$\hat{P}_D^{(k)}(\mathbf{r}, t) = P_{D0}^{(k)} + \hat{P}_b^{(k)}(\mathbf{r}, t) + \hat{P}_d^{(k)}(\mathbf{r}, t) + \hat{P}_{bd}^{(k)}(\mathbf{r}, t) + \hat{P}_{bd}^{(k\dagger)}(\mathbf{r}, t). \quad (29)$$

The latter 4 terms are similar to those of eq.(23) but I will not give their explicit expressions. These 4 terms do not contribute to the mean pressure. The former term of eq.(29) does contribute and, after some algebra, it becomes

$$\begin{aligned}P_{D0}^{(k)} &= \frac{m}{V} \sum_{\mathbf{p}s} \frac{p_k}{E} v_{\mathbf{p}s}^\dagger \alpha_k v_{\mathbf{p}s} = \frac{2}{V} \sum_{\mathbf{p}} \frac{p_k^2}{E} \\ &\rightarrow \frac{1}{\pi^2} \int_0^{p_{\max}} \frac{p_k^2}{E} p^2 dp = \frac{1}{3\pi^2} \int_0^{p_{\max}} \frac{p^4}{E} dp,\end{aligned}$$

where the isotropy has been taken into account in the latter equality. That is the stresses along three orthogonal axes are equal and every one corresponds to the pressure. We get

$$P_{D0} = \frac{1}{12\pi^2} \left[\left(p_{\max}^3 - \frac{3}{2} m^2 p_{\max} \right) \sqrt{m^2 + p_{\max}^2} + \frac{3m^4}{2} \arg \sinh \left(\frac{p_{\max}}{m} \right) \right].$$

The sum, ρ_{ZPF} (for ‘zeropoint field’), of the quantities eqs.(19) and (28) is the energy density of the QED unperturbed vacuum state. It is not obvious whether we should identify the cut-off photon momentum (the same as the energy in natural units), k_{\max} , with either the maximum electron momentum p_{\max} or the maximum electron energy, $E_{\max} \equiv \sqrt{m^2 + p_{\max}^2}$. With both choices the leading term of ρ_{ZPF} becomes

$$\rho_{ZPF} \equiv \rho_{EM0} + \rho_{D0} = -\frac{k_{\max}^4}{8\pi^2} + O(k_{\max}^2), \quad (30)$$

That energy density ρ_{ZPF} is the quantity removed in standard quantum calculations by the normal ordering rule. The sum of the pressures is positive, namely

$$P_{ZPF} \equiv P_{EM0} + P_{D0} = \frac{k_{\max}^4}{8\pi^2} + O(k_{\max}^2). \quad (31)$$

More precisely, neglecting only terms that go to zero when the cut-offs go to infinity, we get

$$\begin{aligned} P_{ZPF} + \rho_{ZPF} &= \frac{1}{6\pi^2} [k_{\max}^4 - p_{\max}^3 E_{\max}] \\ &\quad + \frac{m^4}{4\pi^2} \arg \sinh \left(\frac{p_{\max}}{m} \right), \end{aligned} \quad (32)$$

that suggests the identification

$$k_{\max}^4 = p_{\max}^3 E_{\max}.$$

It is remarkable that the *leading terms give rise to a stress-energy tensor fulfilling the Lorentz invariant relation* $P_{ZPF} = -\rho_{ZPF}$, although every vacuum field alone does not fulfil that relation. On the other hand the sum eq.(32) has *positive pressure and negative energy density* contrary to what we might expect. In section 4 I will propose a solution for this anomaly. The relation eq.(32) corresponds to the neglect of the interaction, to be studied in the next subsection, and it has been calculated only for the particular case of QED. Therefore no strong conclusion may be obtained, but our result suggests that the stress-energy tensor of all interacting fields, might be Lorentz invariant in the vacuum (in Minkowski space).

2.4 Corrections for the interaction

The physical vacuum of QED, $|vac\rangle$, is different from the zeroth order vacuum, $|0\rangle$, studied above. The latter is the eigenvector, with the smallest

eigenvalue, of the unperturbed Hamiltonian $H_0 = H_{EM} + H_D$, see eqs.(17) and (27). The former is the eigenvector of $H = H_{EM} + H_D + H_{int}$ with the smallest eigenvalue, that is taking the interactions into account. Finding $|vac\rangle$ as an exact eigenvector of H is not possible and we should use a perturbation method, that is to calculate it as an expansion in powers of the coupling constant, the positron charge, e . Actually only even powers of e would appear and the result becomes an expansion in powers of the fine structure constant $\alpha \equiv e^2/(4\pi\hbar c) \simeq 1/137$.

The interaction Hamiltonian (or energy) operator may be written, in the Coulomb gauge,

$$\hat{\rho}_{int}(\mathbf{r}, t) = -e\hat{\psi}^\dagger \boldsymbol{\alpha} \hat{\psi} \cdot \hat{\mathbf{A}}. \quad (33)$$

The operators $\hat{\psi}, \hat{\psi}^\dagger$ and $\hat{\mathbf{A}}$ contain two terms each when expanded in plane waves, each term corresponding to an infinite sum. One of these terms has creation operators and the other one annihilation operators (see e. g. eq.(12) for the electromagnetic potential vector). This gives rise to 8 terms for $\hat{\rho}_{int}$, eq.(33). Integrating with respect to \mathbf{r} inside the volume V leads to the interaction Hamiltonian. Only two terms survive and we get

$$\begin{aligned} \hat{H}_{int} = & -e \sum_{\mathbf{p}, \mathbf{q}, \mathbf{k}, s, s', \varepsilon} \frac{m}{V^{3/2} \sqrt{2kEE'}} u_s^\dagger(\mathbf{p}) \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon}_{v_{s'}}(\mathbf{q}) \\ & \times \delta_{\mathbf{p}+\mathbf{q}, \mathbf{k}} \hat{\alpha}_{\mathbf{k}, \varepsilon}^\dagger b_{\mathbf{p}, s}^\dagger d_{\mathbf{q}, s'}^\dagger + h.c., \end{aligned} \quad (34)$$

where $\delta_{\mathbf{p}+\mathbf{q}, \mathbf{k}}$ is a Kronecker's delta, $h.c.$ means Hermitean conjugate and

$$E \equiv \sqrt{p^2 + m^2}, E' \equiv \sqrt{q^2 + m^2}.$$

One of the terms of the Hamiltonian may create triples electron-positron-photon and the other term may annihilate triples. The integral in \mathbf{r} causes that the Hamiltonian eq.(34) is invariant under traslations and rotations, whence it can couple only states with the same total momentum and total angular momentum as shown below, see eqs.(35) and (36).

The physical vacuum, $|vac\rangle$, may be obtained as an expansion in powers of the coupling constant α , but for the calculation of the energy to first order we need $|vac\rangle$ only to order $\alpha^{1/2}$, that is we should not consider states with more than three particles (i. e. one electron, one positron and one photon). We have to that order

$$|vac\rangle = N \left[|0\rangle + \sum_j c_j |j\rangle \right], \quad (35)$$

where $|j\rangle$ represents a state with one photon-electron-positron triple (different j correspond to the particles having either different momenta or spins or both). N is a normalization factor. The coefficients c_j may be got from the interaction Hamiltonian, that is

$$c_j = \frac{\langle j | \hat{H}_{int} | 0 \rangle}{E_0 - E_j}, E_j = \langle j | \hat{H}_0 | j \rangle = V \rho_j. \quad (36)$$

By the properties of H_{int} , see above, the states $|j\rangle$ have the same net momentum and angular momentum than the vacuum, that is zero. However they possess positive energy above the vacuum state $|0\rangle$.

The three-particles state may be got from $|0\rangle$ by the action of creation operators as follows

$$|j\rangle = \hat{a}_{\mathbf{k}\varepsilon}^\dagger \hat{b}_{\mathbf{p}s}^\dagger \hat{d}_{\mathbf{q}s'}^\dagger |0\rangle, j \equiv \{\mathbf{k}, \varepsilon, \mathbf{p}, s, \mathbf{q}, s'\}, \quad (37)$$

with the constraints

$$\mathbf{k} + \mathbf{p} + \mathbf{q} = 0, \varepsilon + s + s' = 0.$$

The unperturbed energy of one of these states is the corresponding eigenvalue of the unperturbed Hamiltonian, that is

$$E_j = k + \sqrt{m^2 + \mathbf{p}^2} + \sqrt{m^2 + \mathbf{q}^2} + E_{ZPF}, \quad (38)$$

where E_{ZPF} is the vacuum energy in the volume V , that is the density ρ_{ZPF} , eq.(30), times the volume V . Corrections of higher order in α would involve a sum similar to eq.(35), but containing more general states, that is states having n triples electron-positron-photon, $n = 1, 2, 3, \dots$. All those states, in addition to having nil momentum and angular momentum, possess the quantum numbers of the unperturbed vacuum state, $|0\rangle$, i. e. zero electric charge (and zero leptonic number). Thus the vacuum state $|vac\rangle$ also possesses those properties.

There are other states defined by a sum similar to eq.(35), but orthogonal to the vacuum state $|vac\rangle$. That is states of the form

$$|vac^{(l)}\rangle = \sum_j c_j^{(l)} |j\rangle,$$

where now $|j\rangle$ represents a general state with n triples electron-positron-photon, $j = 0$ corresponding to the unperturbed vacuum state $|0\rangle$. Thus for clarity I will rewrite the vacuum state eq.(35) in the form (not normalized)

$$|vac\rangle = \sum_j c_j |j\rangle.$$

Amongst the states $|vac^{(l)}\rangle$ orthogonal to $|vac\rangle$, that is fulfilling

$$\sum_j c_j^* c_j^{(l)} = 0,$$

the one with the minimal energy will be named “first excited vacuum state”, labelled with $l = 1$. Similarly the “second excited vacuum state”, $l = 2$, will be the state with minimal energy amongst those orthogonal to both the vacuum state and the first excited vacuum state, and so on.

It is common practice in quantum field theory to name “excitations of the vacuum” the states with particles. For instance a state with one electron or a state with one electron and one photon, etc. However the states here called “excited vacuum states” are a particular kind of vacuum excitations, namely those characterized by having the same quantum numbers of the vacuum (e. g. zero charge if we restrict ourselves to QED) and also zero net momentum and net angular momentum. An interesting question is whether these states appear somewhere in nature. My conjecture is that, with some slight modification, they may be the constituents of the “dark matter”, as will be commented in section 5.

3 Two-point correlations of the vacuum fields

Vacuum fluctuations may be characterized by the two-point correlation of the vacuum stress-energy tensor. Assuming homogeneity and isotropy that tensor contains at most two independent parameters, namely energy density and pressure. However the correlations of the different components of the stress-energy tensor may depend on the direction. E. g. the correlation $\langle T_{xx}(\mathbf{r}_a) T_{yy}(\mathbf{r}_b) \rangle$ may depend on the angles of the vector $\mathbf{r}_a - \mathbf{r}_b$ with the directions X and Y . In this paper I will not calculate the two-point correlations of the vacuum fields in detail. But for illustrative purposes I will get the correlation of the density at equal times, which depends on a single

parameter, namely $|\mathbf{r}_a - \mathbf{r}_b|$. I shall do that in the approximation of zeroth order, that is involving quantum expectations in the unperturbed vacuum state $|0\rangle$, rather than the physical vacuum state $|vac\rangle$.

3.1 Two-point density correlation for free vacuum fields

In this subsection I will work in Minkowski space, but see next subsection for a discussion about the validity of this assumption. Using Cartesian coordinates I shall calculate the correlation eq.(11), that is

$$C(r) \equiv \langle vac | \hat{\rho}(\mathbf{r}_1) \hat{\rho}(\mathbf{r}_2) | vac \rangle - \rho^2, \rho \equiv \langle vac | \hat{\rho} | vac \rangle. \quad (39)$$

The energy density operator is the sum of 3 terms, namely

$$\hat{\rho} = \hat{\rho}_{EM} + \hat{\rho}_D + \hat{\rho}_{int}, \quad (40)$$

that have been discussed in section 2, see eqs.(16) and (23) and (33).

I start proving that a numerical constant added to the energy density operator does not change the value of $C(r)$. In fact, if we substitute $\hat{\rho} + K$ for $\hat{\rho}$ in eq.(39) we get

$$\begin{aligned} C'(r) &\equiv \langle vac | [\hat{\rho}(\mathbf{r}_1) + K] [\hat{\rho}(\mathbf{r}_2) + K] | vac \rangle - \langle vac | [\hat{\rho} + K] | vac \rangle^2 \\ &= C(r) + K [\langle vac | \hat{\rho}(\mathbf{r}_1) | vac \rangle + \langle vac | \hat{\rho}(\mathbf{r}_2) | vac \rangle] - 2K \langle vac | \hat{\rho} | vac \rangle \\ &= C(r), \end{aligned}$$

where we have taken into account that $\langle vac | vac \rangle = 1$ and $\langle vac | \hat{\rho}(\mathbf{r}_1) | vac \rangle$ cannot depend on \mathbf{r} due to the traslational invariance of the vacuum. The result implies that we may ignore the term ρ_{ZPF} , eq.(30), which is equivalent to putting $\rho_{ZPF} = 0$, for the calculation of the two-point correlation $C(r)$. That is from now on I should use eq.(40) ignoring the energy density operators $\hat{\rho}_{EM0}$ and $\hat{\rho}_{D0}$.

The calculation for free fields, that is to zeroth order in the interaction, consists of substituting $|0\rangle$ for $|vac\rangle$ in eq.(28) and neglecting the term $\hat{\rho}_{int}$ in eq.(40). Thus the two-point correlation for free fields (i.e. the unperturbed correlation) is the following

$$\begin{aligned} C_0(r) &= \langle 0 | [\hat{\rho}_{EM}(\mathbf{r}_1) + \hat{\rho}_D(\mathbf{r}_1)] [\hat{\rho}_{EM}(\mathbf{r}_2) + \hat{\rho}_D(\mathbf{r}_2)] | 0 \rangle \\ &= \langle 0 | \hat{\rho}_{EM}(\mathbf{r}_1) \hat{\rho}_{EM}(\mathbf{r}_2) | 0 \rangle + \langle 0 | \hat{\rho}_D(\mathbf{r}_1) \hat{\rho}_D(\mathbf{r}_2) | 0 \rangle, \end{aligned} \quad (41)$$

where the equality is a consequence of the fact that the cross terms $\langle 0 | \hat{\rho}_{EM}(\mathbf{r}_1) \hat{\rho}_D(\mathbf{r}_2) | 0 \rangle$ and $\langle 0 | \hat{\rho}_D(\mathbf{r}_1) \hat{\rho}_{EM}(\mathbf{r}_2) | 0 \rangle$ give no contribution. The electromagnetic free field contribution becomes

$$\begin{aligned} C_{EM0}(r) &= \langle 0 | \hat{\rho}_{EM}(\mathbf{r}_1) \hat{\rho}_{EM}(\mathbf{r}_2) | 0 \rangle = \langle 0 | \hat{\rho}_{EM2}(\mathbf{r}_1) \hat{\rho}_{EM2}(\mathbf{r}_2) | 0 \rangle \\ &= \frac{1}{16V^2} \left\langle 0 \left| \sum_{\mathbf{k}, \varepsilon} \sum_{\mathbf{k}', \varepsilon'} K_2 \hat{\alpha}_{\mathbf{k}, \varepsilon} \hat{\alpha}_{\mathbf{k}', \varepsilon'} \sum_{\mathbf{k}'', \varepsilon''} \sum_{\mathbf{k}''', \varepsilon'''} K_2^* \hat{\alpha}_{\mathbf{k}'', \varepsilon''}^\dagger \hat{\alpha}_{\mathbf{k}''', \varepsilon'''}^\dagger \right| 0 \right\rangle, \end{aligned}$$

where we have taken into account that the term involving $\hat{\rho}_{EM1}(\mathbf{r})$ does not contribute (it annihilates the unperturbed vacuum state $|0\rangle$, see eq.(??)). Hence putting the operators in normal order, using the commutation rules, and introducing an exponential convergence factor we get after some algebra

$$C_{EM0}(r) = \frac{1}{4V^2} \sum_{\mathbf{k}, \varepsilon} \sum_{\mathbf{k}', \varepsilon'} k k' \left[1 + \frac{\mathbf{k} \cdot \mathbf{k}'}{k k'} \right]^2 \exp[i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r} - \varepsilon(k + k')],$$

where eqs.(14) and (15) have been taken into account.

Substituting integrals for the sums this leads to

$$\begin{aligned} C_{EM0}(r) &= \frac{1}{4(2\pi)^6} \int k d^3\mathbf{k} \int k' d^3\mathbf{k}' \left[1 + \frac{\mathbf{k} \cdot \mathbf{k}'}{k k'} \right]^2 \\ &\quad \times \exp[i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r} - \varepsilon(k + k')]. \end{aligned} \quad (42)$$

The exponential convergence factor $\exp[-\varepsilon(k + k')]$ is so chosen in order to get easy integrals, analytical in some cases. It is plausible that the result would not depend dramatically on the type of cut-off used, taking into account that the approximations involved in our calculation will allow only obtaining a rough approximation. The details of the integrals may be seen in the Appendix and the result is

$$C_{EM0}(r) = \frac{3r^4 - 10r^2\varepsilon^2 + 3\varepsilon^4}{\pi^4(\varepsilon^2 + r^2)^6}. \quad (43)$$

The correlation $C_{EM0}(r)$ is positive for small r as it should, because for $r = 0$ the correlation becomes the variance. That is a fluctuation above (below) the average energy density is most probably close to another one also above (below) it. For $r \in (\varepsilon/3, 3\varepsilon)$ the correlation is negative and it is again positive for $r > 3\varepsilon$.

A calculation of the two-point correlation of the free electron-positron field starts from eq.(23), whence we get for $t = 0$

$$\begin{aligned} C_{D0}(r) &= \langle 0 | \hat{\rho}_D(\mathbf{r}_1) \hat{\rho}_D(\mathbf{r}_2) | 0 \rangle = \langle 0 | \hat{\rho}_{bd}(\mathbf{r}_1) \hat{\rho}_{bd}^\dagger(\mathbf{r}_2) | 0 \rangle \\ &= \frac{1}{V^2} \sum_{\mathbf{p}\mathbf{p}'ss'} |F(\mathbf{p}, s, \mathbf{p}', s')|^2 \exp[i(\mathbf{p} + \mathbf{p}') \cdot \mathbf{r}], \end{aligned}$$

where we have taken into account that the terms $\rho_b(\mathbf{r})$ and $\rho_d(\mathbf{r})$ do not contribute because they annihilate the state $|0\rangle$. Hence inserting the expression of $|F|^2$ eq.(24), we obtain

$$\begin{aligned} C_{D0}(r) &= \frac{1}{4V^2} \sum_{\mathbf{p}\mathbf{p}'} (E - E')^2 \left[1 + \frac{\mathbf{p} \cdot \mathbf{q} - m^2}{EE'} \right] \exp[i(\mathbf{p} + \mathbf{p}') \cdot \mathbf{r}] \\ &\rightarrow \frac{1}{4(2\pi)^6} \int d^3\mathbf{p} \int d^3\mathbf{q} (E - E')^2 \left[1 + \frac{\mathbf{p} \cdot \mathbf{q} - m^2}{EE'} \right] \\ &\quad \times \exp[i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} - \varepsilon(p + q)], \end{aligned} \quad (44)$$

after introducing a cut-off in the particles momenta. The integrals are involved and they will not be calculated exactly. If $m \ll \varepsilon^{-1}$ we might approximate $E = \sqrt{p^2 + m^2} \simeq p$, $E' \simeq q$, whence eq.(44) leads to

$$C_{D0}(r) \simeq \frac{3r^4 - 10r^2\varepsilon^2 + 3\varepsilon^4}{2\pi^4(r^2 + \varepsilon^2)^6} + O(m^2). \quad (45)$$

Details of the calculation may be seen in the Appendix.

The contributions of order α to the correlation are straightforward although lengthy. They will not be reported in this paper.

3.2 The relevance of metric fluctuations: A natural cut-off

If spacetime is Minkowski then the two-point energy density correlation of the vacuum fields may depend only on the interval between points that is, with an obvious metric,

$$\langle vac | \hat{\rho}(\mathbf{r}_1, t_1) \hat{\rho}(\mathbf{r}_2, t_2) | vac \rangle = C(\sigma), \sigma^2 \equiv (t_2 - t_1)^2 - |\mathbf{r}_1 - \mathbf{r}_2|^2. \quad (46)$$

This would be a consequence of the Lorentz invariance of the vacuum. Of course the correlation might depend on whether the interval is timelike or spacelike (in other words if σ as defined above is real or imaginary). However *a Minkowski spacetime is not compatible with the existence of fluctuations*. Therefore eq.(46) is at most an approximation.

The proof of incompatibility is trivial and it follows. Let us consider a Minkowski space and two arbitrary points, A and B , with coordinates (\mathbf{r}_A, t_A) and (\mathbf{r}_B, t_B) , respectively. It is always possible to find another point C with coordinates (\mathbf{r}_C, t_C) which is lightlike separated from A and also from B . In fact any point in the intersection of the light cones of A and B fulfils that condition. Then eq.(46) implies that the vacuum energy density in C will be the same as in A and the same as in B . But the points A and B being arbitrary we conclude that the density will be the same in all points of the Minkowski space. There would be no fluctuations at all!

We may arrive at the same conclusion taking into account that if the stress-energy tensor of the vacuum fields fluctuates, then the metric tensor should also fluctuate by Einstein equation of general relativity. The argument is clear in the context of classical physics and it should be taken into account also in the quantum realm. In the calculation of the two-point correlation eq.(41) we have used (implicitly) the Minkowski *classical* metric $diag(1, 1, 1, -1)$, but we should use a *quantum* metric tensor operator $\hat{g}_{\mu\nu}(x)$, where x represents the four coordinates of a point. That operator must be related to the density operator, $\hat{\rho}(x)$, (more generally to the stress-energy tensor operator) via a *quantized Einstein equation*. However a satisfactory quantum gravity theory is not yet available. Therefore a correct calculation of the two-point correlation cannot be made. With our present knowledge it is only possible to perform approximate calculations that mix classical general relativity with quantum theory in some form.

The standard method to deal with problems involving both quantum theory and general relativity is the use of the “semiclassical approximation”. It consists of calculating quantum averages in a given spacetime (usually Minkowski) and putting the averaged quantities on the right side of Einstein equation, this treated as classical, in order to get the spacetime metric. However the semiclassical approximation is not good enough for our purposes in this paper. In fact its use leads to losing two relevant properties of the two-point density correlations of the vacuum fields. Firstly it does not allow deriving the existence of a natural cut-off in the energies of the particles associated to the vacuum fields, as is made in the following. Secondly the

approximation prevents calculating the long range effect on the metric of the vacuum fluctuations, as will be discussed in section 4.

In order to go beyond the semiclassical approximation I will use a different approach that consists of treating the metric fluctuations, actually deriving from the quantum character of the metric, as if they were associated to a classical statistical ensemble of metrics. That is I will replace the correct, fully quantum, vacuum expectation by *a classical average over a statistical distribution of quantum vacuum expectations*, each expectation calculated with a given classical metric.

More explicitly I will consider a probability distribution on an ensemble, J , of metrics, all of them close (in some sense to be specified later) to a reference Mikowski metric $diag(1, 1, 1, -1)$. I will use for every metric of the ensemble J the same (Cartesian) coordinate system as in Minkowski space, but obviously a different metric tensor, $g_{\mu\nu}^j$, and a different volume element. Our problem is to find a more accurate substitute for the calculation made in the previous subsection, i. e. C_0 eq.(41). Specifically the calculation in Minkowski space required integrals involving dynamical variables, like momenta of the particles, multiplied by position vectors, see for instance eqs.(42) or (44). We should find substitutes for those integrals in any metric $g_{\mu\nu}^j$, and also we should find the probability distribution in the ensemble of metrics. The calculation would be involved, or impossible, and I shall make a simplification. I will just consider a family of extremely simple metrics in order to get hints about the main consequences of the metric fluctutations.

Let us consider a family of metrics with the form $diag(\gamma, 1, 1, -1)$, γ being a (positive) real number close to unity, and let us study the change produced by that metric in a typical integral appearing in eq.(42) (where now we remove the cut-off, $\exp[-\varepsilon k]$, because deriving the existence of a cut-off is just the purpose of the present calculation). For instance we will study the change produced in the following integral

$$I \equiv \int k d^3\mathbf{k} \exp[i\mathbf{k} \cdot \mathbf{r}] = \int dk_x \int dk_y \int dk_z |\mathbf{k}| \exp[i\mathbf{k} \cdot \mathbf{r}],$$

where \mathbf{k} is the momentum (or wavevector) of a virtual photon of the vacuum. Actually the new metric, $diag(\gamma, 1, 1, -1)$, corresponds to a Minkowski space too, therefore it is plausible that a calculation in the new metric would lead to the integral

$$I_\gamma \equiv \gamma \int dk_x \int dk_y \int dk_z |\mathbf{k}| \exp[ik_x \gamma x + ik_y y + ik_z z],$$

where the overall factor γ takes the change of the volume element into account. Now let us assume that the distribution of metrics in the family corresponds to a Gaussian probability distribution, $f(\gamma)$, where γ is now taken as a random variable. That is

$$f(\gamma)d\gamma = \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{(\gamma-1)^2}{4\sigma^2}\right) d\gamma,$$

(the probability $f(\gamma)$ is only approximately normalized because the variable γ cannot have negative values). The average of the integrals I_γ in the ensemble of metrics would be

$$\langle I_\gamma \rangle = \int_0^\infty I_\gamma f(\gamma) d\gamma \simeq \int dk_x \int dk_y \int dk_z |\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{r} - \sigma^2 k_x^2 x^2),$$

where with negligible error we have extended the γ integration down to $-\infty$ and approximated γ by unity in the overall factor.

We could extend the family including other metrics like $diag(1, \gamma, 1, -1)$ or $diag(1, 1, \gamma, -1)$. The net result would be that the average value of the integral $\langle I_\gamma \rangle$ should exhibit a cut-off in the photon momenta. This illustrative calculation supports the intuitive idea that randomness implies fuzzyness and fuzzyness causes some average over photon momenta in the exponential $\exp(i\mathbf{k} \cdot \mathbf{r})$. The fuzziness increases with the momentum k , which amounts to an effective ultraviolet momenta cut-off. However neither the calculation nor the intuition allow us finding the form of the cut-off. Actually for the purpose of getting just a rough estimate of the effect of the vacuum fluctuations the form of the cut-off is irrelevant. However an estimate of the maximum allowed particle's momenta (say, an estimate of the parameter σ) is required. Finding such an estimate is difficult, but arguments for its plausible order of magnitude follow.

It is common wisdom that there are quantum metric fluctuations at the Planck scale. If these are the only relevant metric fluctuations then the momentum cut-off should be of order the inverse of the Planck length, which would imply an extremely large value for the two-point correlations (e.g those calculated in eqs.(42) and (44)). That is the energy of the vacuum fluctuations would be of order the vacuum energy itself. It seems more plausible that the cut-off momentum is far smaller, of order the typical particle scales. That is it should be of order the inverse of a Compton wavelength, h/mc , m being a typical (or average) particles mass.

4 Vacuum fluctuations and dark energy

4.1 The need of a cosmological constant in the Einstein equation

It might be that the energy density of the different vacuum fields cancel to each other, and similarly for the pressure, but our calculations within QED, sections 2.2 and 2.3, do not support that assumption. A weaker assumption, consistent with the desired properties of the vacuum, is that the whole set of vacuum fields have a stress-energy tensor proportional to the metric tensor. Furthermore our QED result suggests that this may be the case, but the proportionality constant is divergent, or huge if we assume a natural cut-off at the Planck scale. However this gives rise to the problem commented in the introduction section, namely a big discrepancy between theory and observations. A plausible solution to the problem is proposed in the following.

We return to the discussion of section 1.2 about the relevance of the vacuum fluctuations, but now in the realm of general relativity rather than Newtonian gravity, although still within classical (not quantized) theories. Einstein equation may be written

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = -8\pi G (T_{\mu\nu}^{matt} + T_{\mu\nu}^{vac}), \quad (47)$$

(here I use the notation of the book by Weinberg[9].) The latter term of the left side involves the cosmological constant Λ . On the right side the former term is due to matter and the latter to the vacuum fields, both stress-energy tensors, $T_{\mu\nu}^{matt}$ and $T_{\mu\nu}^{vac}$, depending on spacetime coordinates. The variation of the vacuum tensor takes into account that vacuum fields fluctuate, see section 1-2. We will assume that these fluctuations are short ranged, as discussed at the end of section 3.2.

Our proposed solution is that either the mean vacuum stress-energy tensor is zero or it is precisely balanced by a cosmological constant, whilst the fluctuations give rise to a long range effect able to explain dark energy. If we pass from classical to quantum theory, then vacuum expectations should be substituted for the averages over the volume V . Thus the main assumptions in this paper may be stated as follows:

Proposition 1 The quantum expectation value of the stress-energy tensor operator due to the vacuum fields is proportional to the expectation of the metric tensor (or zero).

That is

$$\langle T_{\mu\nu}^{vac} \rangle \equiv \langle vac | \hat{T}_{\mu\nu}^{vac} | vac \rangle = T^{vac} \langle vac | \hat{g}_{\mu\nu} | vac \rangle, \quad (48)$$

where T^{vac} is a constant and $\hat{g}_{\mu\nu}$ is the quantized metric tensor operator. This implies that (the vacuum expectation of) the stress-energy tensor of the combined vacuum fields has the form of a cosmological constant in the Einstein equation, a hypothesis that is common wisdom. That wisdom however suffers from a big difficulty, namely the said cosmological constant would be huge, which leads us again to the problem discussed in the introduction section, that is the disagreement between eqs.(1) and (2). For me the most plausible solution to the problem is the following assumption:

Proposition 2 There is a cosmological term in the Einstein equation that exactly balances the effect of the mean vacuum stress-energy tensor. That is

$$8\pi G T^{vac} = \Lambda. \quad (49)$$

Assuming that the cancelation is exact we avoid any claim of conspiracy (see section 1.2). Eq.(49) suggests writing the quantized Einstein eq.(47) in the form

$$\hat{G}_{\mu\nu} = -8\pi G \left(\hat{T}_{\mu\nu}^{matt} + \delta\hat{T}_{\mu\nu}^{vac} \right), \delta\hat{T}_{\mu\nu}^{vac} \equiv \hat{T}_{\mu\nu}^{vac} - T^{vac} \hat{g}_{\mu\nu}, \quad (50)$$

where $\hat{G}_{\mu\nu}$, $\hat{T}_{\mu\nu}^{matt}$ and $\delta\hat{T}_{\mu\nu}^{vac}$ are tensor operators. The meaning of this equation is not obvious because there is not yet a satisfactory quantum gravity theory and therefore it is not known how to relate the metric tensor operator $\hat{g}_{\mu\nu}$ with the Einstein tensor operator $\hat{G}_{\mu\nu}$. I shall present in section 4.3 a procedure to give a sense to some approximation of eq.(50) and to get sensible solutions. I will point out that this equation departs from standard semiclassical approach to general relativity, where the source term in a given quantum state should be the expectation value of the relevant field operators in that state. In fact, the semiclassical alternative to eq.(50) would be

$$G_{\mu\nu} = -8\pi G \langle vac | \hat{T}_{\mu\nu}^{matt} + \delta\hat{T}_{\mu\nu}^{vac} | vac \rangle, \quad (51)$$

but the term $\langle vac | \delta\hat{T}_{\mu\nu}^{vac} | vac \rangle$ has been assumed nil, see eq.(48). Consequently the effect of the vacuum fluctuations would be lost in a standard semiclassical treatment.

4.2 The gravity of vacuum fluctuations. A toy model

In order to study the spacetime curvature produced by density and pressure fluctuations we must start from Einstein eq.(47), which leads to the quantized Einstein eq.(50) with our assumptions. Ignoring the effect of matter, it simplifies to

$$\hat{G}_{\mu\nu} = -8\pi G \delta \hat{T}_{\mu\nu}^{vac}, \quad (52)$$

where the Einstein tensor operator $\hat{G}_{\mu\nu}$ is a functional of the metric tensor operator $\hat{g}_{\mu\nu}$. The functional $\hat{G}_{\mu\nu}[\hat{g}_{\lambda\sigma}]$ would contain $\hat{g}_{\lambda\sigma}$ and its first and second derivatives with respect to the coordinates, but we do not know what is the exact relation, something that should be derived from a quantum gravity theory. As a consequence eq.(52) is not well defined. Nevertheless for the study of vacuum fluctuations an approximate quantum equation may be substituted for eq.(52), and sensible solutions found, as shown in the next section.

Before presenting that solution, I will work an extremely simplified model where the two-point density correlations depend on a single parameter, the radial coordinate. Of course the model is far from realistic, but it provides hints about the actual consequences of the vacuum fluctuations. I consider a static problem with spherical symmetry in standard (curvature) coordinates $\{r, \theta, \phi, t\}$ with some stress-energy tensor whose components fulfil

$$\langle vac | (\delta \hat{T}^{vac})_{\mu}^{\nu} | vac \rangle = 0, \quad (53)$$

as a consequence of eq.(48). Using the metric

$$d\sigma^2 = A(r) dr^2 + r^2 [d\theta^2 + \sin^2 \theta d\phi^2] - B(r) dt^2, \quad (54)$$

one of the components of the classical Einstein's equation is[9]

$$\frac{d}{dr} \left(\frac{r}{A} \right) = 1 - 8\pi G r^2 \rho(r), \quad (55)$$

with the initial condition $A(0) = 1$. We shall work in quantized general relativity. Although no satisfactory quantum gravity theory exists yet, it is plausible that the relations between the quantum operators are the same as the classical ones whenever there is no problem with the non-commutativity

of the operators. Therefore the quantized counterpart of eq.(55) relevant for our problem would be

$$\frac{d}{dr} \left(\frac{r}{\hat{A}} \right) = 1 - 8\pi G r^2 \delta\hat{\rho}(r), \quad (56)$$

where $8\pi\delta\hat{\rho}$ is the component of the stress-energy tensor, $(\delta\hat{T}^{vac})^\nu_\mu$, derived from the vacuum fluctuations. I stress that the average tensor due to the vacuum fields does not appear in eq.(?) because I have assumed that it is exactly canceled by a cosmological constant. However for notational simplicity I will substitute $\hat{\rho}$ for $\delta\hat{\rho}$ from now on, but take into account that we will have

$$\langle vac | \hat{\rho} | vac \rangle = 0. \quad (57)$$

The exact solution of this operator equation is trivial and it may be written in closed form, that is

$$\hat{A} = \left[1 - \frac{8\pi G}{r} \int_0^r x^2 \hat{\rho}(x) dx \right]^{-1}. \quad (58)$$

Now we assume that Newton constant G is small (e. g. in comparison with $c\hbar/m^2$, m being a typical particle mass). Thus we expand eq.(58) in powers of G giving

$$\begin{aligned} \hat{A} = & 1 + \frac{8\pi G}{r} \int_0^r x^2 \hat{\rho}(x) dx \\ & + \frac{64\pi^2 G^2}{r^2} \int_0^r x^2 \hat{\rho}(x) dx \int_0^r y^2 \hat{\rho}(y) dy + O(G^3). \end{aligned} \quad (59)$$

The quantized densities at different points may not commute, that is we might have

$$[\hat{\rho}(x), \hat{\rho}(y)] \neq 0,$$

but this is irrelevant in eq.(59) because the latter term is itself the square of an operator. The interesting quantity is the vacuum expectation of the metric, which would provide the observable space-time curvature. The expectation of the above equation leads to

$$\langle vac | \hat{A} | vac \rangle \simeq 1 + \frac{64\pi^2 G^2}{r^2} \int_0^r x^2 dx \int_0^r y^2 dy \langle vac | \hat{\rho}(x) \hat{\rho}(y) | vac \rangle, \quad (60)$$

where eq.(53) has been taken into account. As a conclusion we have obtained the expectation value of one of the metric components in terms of the two-point correlation of the energy density, and *shown that the metric component may have long range (of order r) deviations from Minkowski metric, in spite of the vacuum expectation of the density being zero*, see eq.(57). I stress that this result could not be derived from the standard “semiclassical approximation” to general relativity. In that approximation we should write, instead eq.(56), the following

$$\frac{d}{dr} \left(\frac{r}{\langle vac | \hat{A} | vac \rangle} \right) = 1 - 8\pi G r^2 \langle vac | \delta \hat{\rho}(r) | vac \rangle = 1,$$

whence we would get

$$\langle vac | \hat{A} | vac \rangle = 1,$$

that is the Minkowski value.

We may search for an effective (classical) density, $\rho_{eff}(r)$, providing the same spacetime curvature as the vacuum expectation of the metric component, eq.(60). It would fulfil

$$\langle vac | \hat{A} | vac \rangle = 1 + \frac{8\pi G}{r} \int_0^r x^2 \rho_{eff}(x) dx + O(G^2),$$

that after inserting it in eq.(60) and calculating the derivative leads to

$$\rho_{eff}(r) \simeq \frac{8\pi G}{r^2} \frac{d}{dr} \left[\frac{1}{r} \int_0^r x^2 dx \int_0^r y^2 dy \langle vac | \hat{\rho}(x) \hat{\rho}(y) | vac \rangle \right]. \quad (61)$$

This result emphasizes the main consequence to be derived from our toy model, namely that the vacuum fluctuations produce a metric equivalent to the metric derived from some effective (fictitious) density.

The second component of the classical Einstein equation cannot be quantized as easily as eq.(55), so that we may follow a different route in order to get the quantized component of the metric, $\hat{B}(r)$, see eq.(54). We start from the well known classical solution

$$B(r) = \exp \left[G \int_0^r \frac{2M(x) + 8\pi x^3 P(x)}{x - 2GM(x)} dx \right], \quad (62)$$

where

$$M(x) \equiv \int_0^x 4\pi y^2 \rho(y) dy,$$

P being the pressure (more correctly the component δT_1^1 of the stress-energy tensor eq.(52) divided by -8π). Expanding eq.(62) to second order in G we get

$$\begin{aligned} B(r) = & 1 + G \int_0^r \frac{2M(x) + 8\pi x^3 P(x)}{x} dx + \frac{1}{2} G^2 \left[\int_0^r \frac{2M(x) + 8\pi x^3 P(x)}{x} dx \right]^2 \\ & + 2G^2 \int_0^r \frac{[2M(x) + 8\pi x^3 P(x)]M(x)}{x^2} dx + O(G^3). \end{aligned}$$

In order to quantize this expression we should substitute operators for the dynamical variables. However there is a difficulty due to the possible non-commutativity of $\hat{P}(x)$ and $\hat{M}(x)$. I propose the following rule:

*For the quantization of the product of **two** components of the stress-energy tensor we should use the symetrized product.*

This leads to

$$\begin{aligned} P(x) M(x) &= \int_0^x 4\pi y^2 dy P(x) \rho(y) \\ &\rightarrow \int_0^x 2\pi y^2 dy \left[\hat{P}(x) \hat{\rho}(y) + \hat{\rho}(y) \hat{P}(x) \right] \\ &= \frac{1}{2} \left[\hat{P}(x) \hat{M}(x) + \hat{\rho}(y) \hat{M}(x) \right]. \end{aligned}$$

Then the vacuum expectation of the quantized metric component $\hat{B}(r)$ becomes

$$\begin{aligned} \langle vac | \hat{B} | vac \rangle &= 1 + 4G^2 \int_0^r \frac{\langle vac | [\hat{M}(x)]^2 | vac \rangle}{x^2} dx \\ &+ \frac{1}{2} G^2 \left\langle vac \left| \left[\int_0^r \frac{2\hat{M}(x) + 8\pi x^3 \hat{P}(x)}{x} dx \right]^2 \right| vac \right\rangle \\ &+ 8\pi G^2 \int_0^r \langle vac | [\hat{P}(x) \hat{M}(x) + \hat{M}(x) \hat{P}(x)] | vac \rangle x dx. \end{aligned} \tag{63}$$

We may introduce an effective pressure, $P_{eff}(r)$, such that

$$\begin{aligned}\langle vac | \hat{B} | vac \rangle &= 1 + \exp \left[G \int_0^r \frac{2M_{eff}(x) + 8\pi x^3 P_{eff}(x)}{x - 2GM_{eff}(x)} dx \right] \\ &= 8\pi G \int_0^r \left[P_{eff}(x) + \ln \left(\frac{r}{x} \right) \rho_{eff}(x) \right] x^2 dx + O(G^2)\end{aligned}\quad (64)$$

It is straightforward to get $P_{eff}(r)$ equating this with eq.(63), but I will not write the result.

4.3 Approximate solution in four dimensions

In order to devise a method for the solution of eq.(50), ignoring the effect of matter, I propose the following generalization of the method used in the previous toy model.

We should write the metric tensor, g_{jk} , as an expansion in powers of the Newton constant, G , that may be taken as a small parameter whence the deviation from the Minkowski metric is assumed small. Therefore we write

$$g_{jk} = g_{jk}^{(0)} + Gg_{jk}^{(1)} + G^2g_{jk}^{(2)} \dots$$

Thus the zeroth order approximation of eq.(??) becomes

$$G_{\mu\nu} [g_{jk}^{(0)}] = 0,$$

whence $g_{jk}^{(0)}$ is the metric tensor of Minkowski space. The first order approximation will be

$$-8\pi\delta T_{\mu\nu}^{vac0}(x) = \left(\frac{\delta G_{\mu\nu} [g_{jk}]}{\delta g_{jk}} \right) g_{jk}^{(1)},$$

where the functional derivative should be taken at $g_{jk} = g_{jk}^{(0)}$. The point is that this equation is *linear* in the unknown, therefore relatively easy to solve. The second order approximation would involve the second functional derivative, giving another linear equation, and so on. In summary we reduce the Einstein nonlinear equation to an infinite set of linear ones. Of course this is only valid because we are assuming here that the metric is close to Minkowski, with small fluctuations superimposed on it. As explained below the second order is sufficient for the study of the vacuum fluctuations.

An advantage of the approximate method to solve Einstein eq.(??) is that it may be used, with plausible assumptions, for a treatment in *quantized*

general relativity. In the quantized theory the tensors $T_{\mu\nu}^{vac}$ and g_{jk} become operators, but their relation cannot be obtained from the classical (Einstein) equation due to the possible non-commutativity. However, the approximate relations between the metric and the stress-energy tensors, involved in the calculation up to second order in G , are either linear or they contain products of no more than two components of the stress-energy operator. In the linear relations there is no problem of commutativity and in relations involving products of two operators it is plausible to use the symmetrized product. That is if $T_{\mu\nu}^{vac}(x) T_{\lambda\sigma}^{vac}(x')$ is a classical product, the quantum counterpart would be

$$1/2 \left[\hat{T}_{\mu\nu}^{vac}(x) \hat{T}_{\lambda\sigma}^{vac}(x') + \hat{T}_{\lambda\sigma}^{vac}(x') \hat{T}_{\mu\nu}^{vac}(x) \right],$$

where x represents the four coordinates of a spacetime point.

The conclusions of this section are the following:

- 1) It is possible to quantize the Einstein equation of general relativity in the post-Newtonian approximation, that is neglecting terms of order higher than G^2 provided we assume that the quantum counterpart of the product of two components of the stress-energy tensor gives a symmetrized product of operators.
- 2) The vacuum expectation of the metric induced by the quantum vacuum fluctuations is the same as the one due to a classical effective stress-energy tensor, that is an effective mass density, ρ_{eff} , and pressure, P_{eff} if the spacetime is Minkowski (at least with good enough approximation).
- 3) The effective mass density is given by Newton constant, G , times a quantity which may depend only on the masses of the particles involved (i. e. the electron mass if we study just QED effects) and the universal constants.

4.4 An estimate of the dark energy density

In the previous sections of this paper I have sketched a research program that in principle would allow calculating the properties of dark energy in terms of the fundamental quantum fields. I exclude the possibility that dark energy may be identified with the mean stress-energy of the vacuum fields, which would give an unplausibly huge value. In fact I assume that a cosmological constant exists in Einstein equation that exactly balances that mean field energy, but I suppose that the cosmological constant cannot balance the energy of the vacuum fluctuations. Thus a calculation of the dark energy

properties should start from the study of the two-point correlations of the fluctuations of the vacuum stress-tensor. It is shown that the result would be finite due to metric fluctuations (see section 3). The vacuum expectation of the two-point correlation gives rise to a long range spacetime curvature as suggested in subsection 4.3, which would allow in principle getting the effective stress-energy tensor. It is straightforward to get a classical effective stress-energy tensor providing the same curvature as the two-point quantum correlation. Finally we should identify that effective mass density with the density, eq.(2), of the dark energy thus getting the properties of dark energy from fundamental quantum fields.

Actually a calculation along the lines of the previous subsection, involving a simplified static model (i. e. a model in 3 rather than 4 dimensions), has been worked elsewhere[6]. The model is more realistic than the one of section 4.2 in the sense that it does not assume spherical symmetry. That is the two-point correlations of the stress-energy tensors depend on the spatial distance between the points, rather than on the difference between the radial coordinates as in the model of section 4.2. However the main conclusion is similar as in ours eqs.(61) and (64). Namely a stress-energy tensor operator with nil vacuum expectation gives rise to a long range spacetime curvature equivalent to the one produced by an effective (classical) stress-energy tensor. That tensor is proportional to some integral of the two-point correlation $C(r)$. In particular with some plausible assumptions the said model[6] gives the following effective density, ρ_{eff} , and pressure, P_{eff} ,

$$\rho_{eff} = -P_{eff} \sim KG \int_0^\infty C(r) r dr, \quad (65)$$

where the numerical constant K in front of the integral is of order unity. It is interesting that ρ_{eff} so obtained roughly equals the gravitational energy of the two-point correlations if calculated with Newtonian gravity, eq.(5). If the vacuum fluctuations are the cause of the dark energy we should identify ρ_{eff} in eq.(65) with ρ_{DM} in eq.(2) whence we get the following estimate for the QED contribution to the dark energy density

$$\rho_{DM} \sim KG \int_0^\infty C(r) r dr \sim G\epsilon^{-6} [1 + O(m^2\epsilon^2)], \quad (66)$$

where $C(r)$ is identified our calculated value in eqs.(43) and (45).

This is the QED contribution to the dark energy density, that is the contribution of the electromagnetic and electron-positron vacuum fields (calculated

to zeroth order in the perturbation). It is plausible that the contribution of all quantum fields would be of order eq.(66) times a few tens. In any case the main obstacle for a valid estimate of the dark energy density is the difficulty to calculate the cut-off derived from the metric fluctuations, as discussed in subsection 3.2. Before solving this problem it is not worth to refine the calculations.

A guess about the value of the said cut-off may be taken as follows. The parameter ε was introduced in section 3.1 as the inverse of the maximum particle's momentum and we have argued in section 3.2 that a plausible estimate should be of order of some average particle mass, m , times c . Thus we get

$$\rho_{DM} = \rho_{eff} \sim \frac{Gm^6c^2}{\hbar^4},$$

where we have written explicitly the velocity of light and Planck's constant for clarity. That quantity agrees with the observed value, eq.(2), if we put m about 80 times the electron mass, which is not unplausible.

5 Excited vacuum states and dark matter

In section 2.4 we have studied the physical vacuum state, defined as the ground state of the total Hamiltonian, H , of the interacting quantum fields, that is the eigenstate with the smallest eigenvalue. H should include all quantum fields of nature together with their interactions, but in this paper we have used the QED Hamiltonian as an example. It is a straightforward prediction of quantum theory that other eigenstates of H are possible in addition to the ground state. Amongst these states there is a particular class, studied in section 2.4, that I have named *excited vacuum states (EVS)*. We may ask whether those states may be observed somewhere and I propose that they may be candidates for dark matter.

The relevant property is that *EVS* consist of coupled particles possessing the same quantum numbers of the vacuum, that is no charges (either electric, baryonic, leptonic, etc.), and also no net momentum or angular momentum. As a consequence they would not interact via electromagnetic, weak or strong nuclear forces. However such states possess energy above the vacuum, see eq.(38). In section 2.5 I have assumed that the total energy of the vacuum state is exactly balanced by a cosmological constant, but the *EVS* possess

energy above the vacuum and therefore these states should interact gravitationally.

In section 2.4 we studied *EVS* in a Minkowski space, that is in absence of gravitational fields, therefore having translational and rotational invariance. However it is plausible that in the presence of gravitational fields of baryonic matter the states are modified, suffering the attraction of large masses. A more detailed study would be needed in order to know the possible gravitational modification of the *EVS*. In the meantime I conjecture that the effect would be relevant only for extended gravitational fields like those of galaxies and clusters. In any case the *EVS* have properties not too different from those usually ascribed to the hypothetical constituents of dark matter, that is they are cold dark matter, as they possess energy but no pressure (or very small). A more detailed study of the possibility that dark matter consists of *EVS* will not be made in this paper, but I believe that such study is worth to be made.

6 Conclusions

The energy distribution in the quantum vacuum has been studied for the particular case of quantum electrodynamics (QED). It has been shown that the stress-energy tensor of the combined electromagnetic and electron-positron fields is roughly proportional to the metric tensor, but the proportionality constant diverges. As a plausible solution it is proposed that there is a cosmological constant in Einstein equation that exactly balances the stress-energy of the quantum vacuum. Arguments have been given showing that vacuum fluctuations give rise to a metric able to explain dark energy. Finally it has been suggested that dark matter may be an effect of vacuum excitations.

7 Appendix. Two-point correlation to zeroth order

The free electromagnetic field

We shall solve the integrals in eq.(42), that is

$$C_{EM}(\mathbf{r}) = \frac{1}{256\pi^6} \int d^3\mathbf{k} \int d^3\mathbf{k}' k k' \left[1 + \frac{\mathbf{k} \cdot \mathbf{k}'}{k k'} \right]^2 \\ \times \exp [i (\mathbf{k} + \mathbf{k}') \cdot \mathbf{r} - \varepsilon k - \varepsilon k'] .$$

In spherical coordinates with polar angle in the direction of \mathbf{r} we get

$$C_{EM}(\mathbf{r}) = \frac{1}{256\pi^6} \int_0^\infty k^3 dk \int_0^\infty k'^3 dk' \int_{-1}^1 d(\cos \theta) \int_{-1}^1 d(\cos \theta') \\ \times \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \exp(ikr \cos \theta - \varepsilon k) \exp(ik'r \cos \theta' - \varepsilon k') \\ \times [1 + \sin \theta \cos \phi \sin \theta' \cos \phi' + \sin \theta \sin \phi \sin \theta' \sin \phi' + \cos \theta \cos \theta']^2 .$$

Integration of the angles ϕ and ϕ' leads to

$$C_{EM}(\mathbf{r}) = \frac{1}{128\pi^4} \int_0^\infty k^3 dk \int_0^\infty k'^3 dk' \int_{-1}^1 du \int_{-1}^1 du' \\ \times [3 + 3u^2 u'^2 - u^2 - u'^2 + 4uu'] \exp(ikru - \varepsilon k) \exp(ik'ru' - \varepsilon k') ,$$

where $r \equiv |\mathbf{r}|$, $u \equiv \cos \theta$, $u' \equiv \cos \theta'$. In terms of the integrals to be defined below we get

$$C_{EM}(\mathbf{r}) = \frac{1}{32\pi^4} [3I_3^2 + 3I_{3uu}^2 - 2I_3 I_{3uu} + 4I_{3u}^2] \\ = \frac{3r^4 - 10r^2\varepsilon^2 + 3\varepsilon^4}{\pi^4 (r^2 + \varepsilon^2)^6}$$

The free electron-positron field

We shall start from eq.(44), that is

$$C_D(\mathbf{r}) = \frac{1}{256\pi^6} \int d^3\mathbf{p} \int d^3\mathbf{q} (E - E')^2 \left[1 + \frac{\mathbf{p} \cdot \mathbf{q} - m^2}{EE'} \right] \\ \times \exp [i (\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} - \varepsilon (p + q)] , \quad (67)$$

that in polar coordinates becomes

$$C_D(\mathbf{r}) = \frac{1}{64\pi^4} \int_0^\infty p^2 dp \int_0^\infty q^2 dq \int_{-1}^1 du \int_{-1}^1 du' \\ \times (E - E')^2 \left[1 + \frac{pq u u' - m^2}{EE'} \right] \\ \times \exp [i (pu + qu') r - \varepsilon (p + q)]$$

If we approximate

$$E \simeq p + \frac{m^2}{2p}, E' \simeq p + \frac{m^2}{2p},$$

valid for $m \ll \varepsilon^{-1}$, we get to order $O(m^2)$

$$\begin{aligned} & (E - E')^2 \left[1 + \frac{pq uu' - m^2}{EE'} \right] \\ & \simeq (p - q)^2 (1 + uu') - 2m^2 \frac{(p - q)^2}{pq} + \frac{m^2}{2} uu' \frac{(p - q)^4}{p^2 q^2}. \end{aligned}$$

The integrals that appear in the terms of order m^2 are lengthy, although straightforward, and they will not be reported here. The term of zeroth order is

$$\begin{aligned} C_{D0}(r) & \simeq \frac{1}{8\pi^4} [I_2 I_4 - I_3^2 + I_{2u} I_{4u} - I_{3u}^2] + O(m^2) \\ & = \frac{3r^4 - 10r^2 \varepsilon^2 + 3\varepsilon^4}{2\pi^4 (r^2 + \varepsilon^2)^6} + O(m^2). \end{aligned}$$

The integrals

The required integrals are

$$\begin{aligned} I_n & \equiv \frac{1}{2} \int_0^\infty k^n dk \exp(-\varepsilon k) \int_{-1}^1 du \cos(kru), \\ I_{nu} & \equiv \frac{i}{2} \int_0^\infty k^n dk \exp(-\varepsilon k) \int_{-1}^1 u du \sin(kru), \\ I_{nuu} & \equiv \frac{1}{2} \int_0^\infty k^n dk \exp(-\varepsilon k) \int_{-1}^1 u^2 du \cos(kru). \end{aligned}$$

The results for the relevant integrals are as follows

$$\begin{aligned} I_2 & = \frac{2\varepsilon}{(\varepsilon^2 + r^2)^2}, I_3 = \frac{6\varepsilon^2 - 2r^2}{(\varepsilon^2 + r^2)^3}, I_4 = \frac{24\varepsilon(\varepsilon^2 - r^2)}{(\varepsilon^2 + r^2)^4}, \\ I_{2u} & = \frac{2ir}{(\varepsilon^2 + r^2)^2}, I_{3u} = \frac{8ir\varepsilon}{(\varepsilon^2 + r^2)^3}, \\ I_{4u} & = -8ir \frac{r^2 - 5\varepsilon^2}{(\varepsilon^2 + r^2)^4}, I_{3uu} = \frac{2\varepsilon^2 - 6r^2}{(\varepsilon^2 + r^2)^3}. \end{aligned}$$

References

- [1] P. J. E. Peebles y Bharat Ratra, *Rev. Mod. Phys.* **75**, 559-606 (2003).
- [2] Varun Sahni, *Lect. Notes Phys.* **653**, 141-180 (2004).
- [3] P. A. R. Ade et al., Planck Colaboration, Arxiv/ 1303.5062, *Astronomy and Astrophysics* (2013).
- [4] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
- [5] B. Y. Zeldovich, *Sov. Phys. Usp.* **24**, 216 (1981).
- [6] E. Santos, *Astrophys. Space Sci.* 332, 423-435 (2011).
- [7] J. J. Sakurai, *Advanced quantum mechanics*, Addison-Wesley, Reading (Massachusetts), 1967.
- [8] S. S. Schweber, *An introduction to relativistic quantum field theory*, Harper and Row, New York, 1962.
- [9] S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, New York, 1972.